# Notes on wave optics

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## The wave equation

- Because light exhibits wave-particle duality, wave-based descriptions of light are often appropriate in optical physics, allowing the establishment of an electromagnetic theory of light.
- As electric fields can be generated by time-varying magnetic fields and magnetic fields can be generated time-varying electric fields, electromagnetic waves are perpendicular oscillating waves of electric and magnetic fields that propagate through space. For lossless media, the **E** and **B** field waves are in phase.
- By manipulating Maxwell's equations of electromagnetism, two relatively concise vector expressions that describe the propagation of electric and magnetic fields in free space are found. Recall that the constants $\varepsilon_0$ and $\mu_0$ are the permittivity and permeability of free space respectively.

$$\nabla^2 \boldsymbol{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \qquad \nabla^2 \boldsymbol{B} = \varepsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2}$$

- Since an electromagnetic wave consists of perpendicular electric and magnetic waves that are in phase, light can be described using the wave equation (which is equivalent to the expressions above). Note that the speed of light $c = (\varepsilon_0\mu_0)^{-1/2}$. Electromagnetic waves represent solutions to the wave equation.

$$\nabla^2 \boldsymbol{U} = \frac{1}{c^2} \frac{\partial^2 \boldsymbol{U}}{\partial t^2}$$

- Either the electric or the magnetic field can used to represent the electromagnetic wave since they propagate with the same phase and direction. With the exception of the wave equation above, the electric field E will instead be used to represent both waves. Note that either the electric or magnetic field can be employed to compute amplitudes.

## Solutions to the wave equation

- Plane waves represent an important class of solutions to the wave equation. The parameter **k** is the wavevector (which points in the direction of the wave's propagation) with a magnitude equal to the wavenumber $2\pi/\lambda$. In a 1-dimensional system, the dot product **k•r** is replaced by kx. The parameter ω is the angular frequency 2πf and φ is a phase shift.

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \cos(\omega t - \boldsymbol{k} \bullet \boldsymbol{r} + \varphi)$$

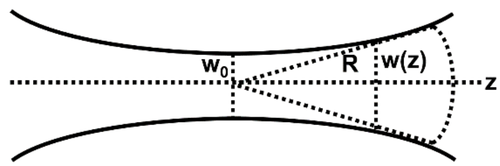
- To simplify calculations, Euler's formula can be used to convert the equation above into complex exponential form. Only the real part describes the wave as the real part corresponds to the cosine term.

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0 \mathrm{Re}\left(e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r} + \varphi)}\right)$$

- Spherical waves are another useful solution to the wave equation (though they are an approximation and truly spherical waves cannot exist). Because of their geometry, the electric field of a spherical wave is only dependent on distance from the origin. As such, the equation for a spherical wave can be written as seen below with origin $\mathbf{r}_0$.

$$U(\boldsymbol{r}) = \frac{U_0}{|\boldsymbol{r} - \boldsymbol{r}_0|} e^{ik|\boldsymbol{r} - \boldsymbol{r}_0|}$$

- Gaussian beams are a solution to the wave equation that can be used to model light from lasers or light propagating through lenses. If a Gaussian beam propagates in the z direction, then from the perspective of the xy plane, it shows a Gaussian intensity distribution. For a Gaussian beam, the amplitude decays over the direction of propagation according to some function A(z), R(z) represents the radius of curvature of the wavefront, and w(z) is the radius of the wave on the xy plane at distance z from the emitter. Often these functions can be approximated as constants.




$$\boldsymbol{E}(x,y,z) = A(z)\left(e^{-\frac{(x^2+y^2)}{w(z)^2}}\right)\left(e^{\frac{ik(x^2+y^2)}{2R}}\right)$$

**Intensity and energy of electromagnetic waves**

- The Poynting vector **S** is oriented in the direction of a wave's propagation (assuming that the wave's energy flows in the direction of its propagation).

$$\boldsymbol{S} = c^2\varepsilon_0 \boldsymbol{E} \times \boldsymbol{B} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}$$

- The magnitude of the Poynting vector represents the power per unit area (W/m$^2$) or intensity crossing a surface with a normal parallel to **S**. Note that this is an approximation since, according to a quantum mechanical description of electromagnetic waves, the energy should be quantized.

$$I = |\boldsymbol{S}| = |c^2\varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}|$$

- Power per unit area (intensity) of plane waves, spherical waves, and Gaussian beams can also be calculated using the equations below. The formula for the

Gaussian beam's power represents the power at a plane perpendicular to the direction of light propagation z.

$$I_{\text{plane}} = \frac{c\varepsilon_0}{2}|\boldsymbol{E}_0|^2, \qquad I_{\text{spherical}} = \frac{|\boldsymbol{E}_0|^2}{|\boldsymbol{r}-\boldsymbol{r}_0|^2}, \qquad I_{\text{Gaussian}} = I_0\left(\frac{w_0}{w(z)}\right)^2 e^{-\frac{2x^2+2y^2}{w(z)^2}}$$

- For electromagnetic waves, instantaneous energy per unit area is difficult to measure, so the average energy per unit area over a period of time Δt is often worked with instead. Since waves are continuous functions, taking their time-average requires an integral.

$$\langle f(t)\rangle_{\Delta t} = \frac{1}{\Delta t}\int_{t-\Delta t/2}^{t+\Delta t/2} f(t)dt$$

- After using the above integral on the function $e^{i\omega t}$ and then taking the real and imaginary parts of the result, the time-averages of the functions cos(ωt) and sin(ωt) are found.

$$\langle\cos(\omega t)\rangle_{\Delta t} = \left(\frac{\sin(\omega\Delta t/2)}{\omega\Delta t/2}\right)\cos(\omega t), \qquad \langle\sin(\omega t)\rangle_{\Delta t} = \left(\frac{\sin(\omega\Delta t/2)}{\omega\Delta t/2}\right)\sin(\omega t)$$

**Superposition of waves**

- Let two waves $E_1$ and $E_2$ of the same frequency traveling in the same direction undergo superposition. $E_1$ and $E_2$ may or may not possess the same amplitude or phase. The substitution α = –(kx+φ) will be carried out.

$$E_1(x,t) = E_{01}\cos\big(\omega t - (kx+\varphi_1)\big) = E_{01}\cos(\omega t - \alpha_1)$$

$$E_2(x,t) = E_{02}\cos\big(\omega t - (kx+\varphi_2)\big) = E_{02}\cos(\omega t - \alpha_2)$$

$$E_1(x,t) + E_2(x,t) = E_{01}\cos(\omega t - \alpha_1) + E_{02}\cos(\omega t - \alpha_2)$$

- If the phases of the waves are different, some special equations are necessary to find the amplitude $E_0$ and the phase α of the resulting wave.

$$E_0 = \sqrt{E_{01}^2 + E_{02}^2 + E_{01}E_{02}\cos(\alpha_2 - \alpha_1)}$$

$$\alpha = \arctan\left(\frac{E_{01}\sin(\alpha_1) + E_{02}\sin(\alpha_2)}{E_{01}\cos(\alpha_1) + E_{02}\cos(\alpha_2)}\right)$$

- For the superposition of any number of waves, the equations above can be extended.

$$E = E_0 \cos(\alpha \pm \omega t) = \sum_{i=1}^{n} E_{0i} \cos(\alpha_i \pm \omega t)$$

$$E_0 = \sqrt{\sum_{i=1}^{N} E_{0i}^2 + 2\sum_{j>1}^{N}\sum_{i=1}^{N} E_{0i}E_{0j}\cos\left(\alpha_i - \alpha_j\right)}$$

$$\alpha = \arctan\left(\frac{\sum_{i=1}^{N} E_{0i}\sin(\alpha_i)}{\sum_{i=1}^{N} E_{0i}\cos(\alpha_i)}\right)$$

**Polarization of light**

- The waves comprising linearly polarized light are all oriented at the same angle which is defined by the direction of the electric field of the light waves. For linearly polarized plane waves with electric fields oriented along the x or y axes that propagate in the z direction, the following equations describe their electric fields.

$$\boldsymbol{E}_x(z,t) = \boldsymbol{\hat{\imath}} E_{x0}\cos(\omega t - kz + \varphi)$$

$$\boldsymbol{E}_y(z,t) = \boldsymbol{\hat{\jmath}} E_{y0}\cos(\omega t - kz + \varphi)$$

- The superposition of two linearly polarized plane waves that are orthogonal to each other (and out of phase) is the vector sum of each electric field.

$$\boldsymbol{E}(z,t) = \boldsymbol{E}_x(z,t) + \boldsymbol{E}_y(z,t)$$

- The superposition of two linearly polarized plane waves that are orthogonal to each other (and in phase) is computed via the following equation and has a tilt angle θ determined by the ratio of amplitudes of the original waves. This process can also be performed in reverse with a superposed polarized wave undergoing decomposition into two orthogonal waves.

$$\boldsymbol{E}(z,t) = \left(\boldsymbol{\hat{\imath}} E_{0x} + \boldsymbol{\hat{\jmath}} E_{0y}\right)\cos(\omega t - kz)$$

$$\tan(\theta) = \frac{E_{0y}}{E_{0x}}$$

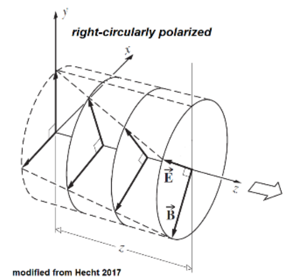
- When two constituent waves possess equal amplitudes and a phase shift of nπ/2, the superposed wave is circularly polarized (as it can be expressed using a sine and a cosine term). Equations for the constituent waves and the superposed wave are given below.

$$\boldsymbol{E}_x(z,t) = \hat{\boldsymbol{\imath}} E_0 \cos(\omega t - kz)$$

$$\boldsymbol{E}_y(z,t) = \hat{\boldsymbol{\jmath}} E_0 \cos(\omega t - kz - \pi/2)$$

$$\boldsymbol{E}(z,t) = \hat{\boldsymbol{\imath}} E_0 \cos(\omega t - kz) + \hat{\boldsymbol{\jmath}} E_0 \sin(\omega t - kz) = E_0 \begin{bmatrix} \cos(\omega t - kz) \\ \sin(\omega t - kz) \end{bmatrix}$$

- When circularly polarized light propagates, it takes a helical path and so rotates. As such, a full rotation occurs after one wavelength. If a circularly polarized wave rotates clockwise, it is called right-circularly polarized and has a positive sine term. If a circularly polarized wave rotates counterclockwise, it is called left-circularly polarized and has a negative sine term.



modified from Hecht 2017

$$\boldsymbol{E}_{\text{right}}(z,t) = E_0 \begin{bmatrix} \cos(\omega t - kz) \\ \sin(\omega t - kz) \end{bmatrix}, \qquad \boldsymbol{E}_{\text{left}}(z,t) = E_0 \begin{bmatrix} \cos(\omega t - kz) \\ -\sin(\omega t - kz) \end{bmatrix}$$

- If a right-circularly polarized light wave and a left-circularly polarized light wave of equal amplitude are superposed, then they create a linearly polarized light wave with twice the amplitude of the individual waves.

$$E_x(z,t) = 2\hat{\boldsymbol{\imath}} E_0 \cos(\omega t - kz)$$

- Linearly polarized and circularly polarized light are special cases of elliptically polarized light. For elliptically polarized light, the amplitudes of the superposed waves may differ and the relative phase shift does not need to be nπ/2. As such, the electric field traces an elliptical helix as it propagates along the z direction.

$$\boldsymbol{E}(z,t) = \hat{\boldsymbol{\imath}} E_{0x} \cos(\omega t - kz) + \hat{\boldsymbol{\jmath}} E_{0y} \cos(\omega t - kz + \varphi)$$

- For elliptically polarized light with a positive phase shift φ, it is called right-elliptically polarized if $E_{0x} > E_{0y}$ and left-elliptically polarized if $E_{0x} < E_{0y}$.
- Most light is unpolarized (or more appropriately, a mixture of randomly polarized waves). To obtain polarized light, polarizing filters are often used.

## References

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