Notes on fiber biomechanics

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Elastic fiber models

- For an elastic fiber in which a linear relationship between force and change in length is assumed, the force is given by $F = k(L - L_0)$.
- To normalize for other elastic fibers with different starting lengths, this equation is divided by L_0 to give $F = k(L/L_0 - 1)$. It is common practice to represent L/L_0 as a parameter λ (called the stretch ratio).
- As such, F is found using the formula below. Note that the quantity λ – 1 is referred to as the strain.

$$F = k(\lambda - 1)$$

- While linear models are often useful, many real fibers exhibit finite extensibility (a nonlinear phenomenon) after exceeding a certain critical strain value λ_c. That is, the force necessary to extend the fiber farther after exceeding λ_c increases rapidly. Finite extensibility can be modeled using the following equation which divides k by a term dependent on λ and λ_c.

$$F = \frac{k(\lambda - 1)}{1 - \frac{\lambda - 1}{\lambda_c - 1}} = \frac{k(\lambda - 1)(\lambda_c - 1)}{\lambda_c - \lambda}$$

- To model a muscle, let L_0 represent the muscle's length in its inactive state and $L_{contracted}$ represent the muscle's length in its contracted state. Unlike the spring, the contracted state is used as the reference length. The contraction stretch is described by the ratio $\lambda_{contracted} = L_{contracted}/L_0$ while the stretch ratio remains as $\lambda = L/L_0$.

$$F = k\left(\frac{\lambda}{\lambda_{\text{contracted}}} - 1\right)$$

- If this muscle is contracted without carrying a load such that F = 0, then $\lambda = \lambda_{contracted}$. If the muscle acquires a load and so must maintain a constant length equal to its original length L_0 (to "hold the load steady"), then the force in the muscle is $F = k(1/\lambda_{contracted} - 1)$.
- To generalize this model for 3-dimensional space, the locations of the fiber's endpoints A and B are used. The fiber's length and orientation are given below.

$$L = |\boldsymbol{x}_B - \boldsymbol{x}_A|, \qquad \boldsymbol{a} = \frac{\boldsymbol{x}_B - \boldsymbol{x}_A}{|\boldsymbol{x}_B - \boldsymbol{x}_A|}$$

- The following force vectors can act on point B and on point A. The stretch ratio is still $\lambda = L/L_0$.

$$\boldsymbol{F}_B = k(\lambda - 1)\boldsymbol{a}, \qquad F_A = -k(\lambda - 1)\boldsymbol{a}$$

Viscous fiber models

- Purely viscous behavior (as with liquids) can be described 1-dimensionally using the equation below where c_η is a damping coefficient.

$$F = \frac{c_\eta}{L}\frac{dL}{dt}$$

- The normalized rate of deformation is equivalent to the above formula without the damping coefficient. In addition, the rate of deformation can be written in terms of the stretch ratio $\lambda = L/L_0$.

$$D = \frac{1}{L}\frac{dL}{dt} = \frac{1}{\lambda}\frac{d\lambda}{dt}$$

- If one endpoint of a filament of fluid is moved with constant velocity, its position is given by $x_B = L_0 + vt$. This means that the rate of deformation is v/x_B.

$$D = \frac{1}{L}\frac{dL}{dt} = \frac{v}{L_0 + vt}$$

- Solving the above equation gives the following result. For a constant rate of elongation, the point x_B must be displaced exponentially over time.

$$L = L_0 e^{Dt}$$

- Given endpoint displacements u_A and u_B, the total displacement is $\Delta L = u_B - u_A$. Using this quantity, the stretch λ and the strain ε can be written using the equations below.

$$\lambda = \frac{L_0 + \Delta\mathrm{L}}{L_0} = 1 + \varepsilon, \qquad \varepsilon = \frac{\Delta L}{L_0}$$

- For the small strains (i.e. $|\varepsilon|$ is much less than 1) that result from small stretches, some approximations can be made which reduce the force equation to the following form.

$$F = c_\eta \frac{d\varepsilon}{dt}$$

Viscoelastic fiber models

- Many biological materials exhibit viscoelastic behavior rather than elastic behavior. In viscoelastic systems, the force on a fiber with a constant length decreases over time and applying constant force causes the length to increase.
- The strain response of a fiber to force is given as $\varepsilon(t)$. The creep function $J(t)$ describes the fiber's tendency to permanently deform as strain is applied. Many possible creep functions can be devised depending on the system. The creep function is related to the strain by a factor of force increase F_0.

$$\varepsilon(t) = J(t)F_0$$

- Strain responses follow the principle of superposition. That is, if a force is applied at τ_1 and then another force is applied at τ_2, a total strain response can be expressed as a sum of the two individual strain responses.

$$\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) = J(t - \tau_1)F_1 + J(t - \tau_2)F_2$$

- For an arbitrary history of applied forces, an integral formulation of strain response is used. In this equation, the change in force at each infinitesimal time interval is multiplied by the creep function.

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\frac{dF}{dt}d\tau$$

- Similarly, the force resulting from an imposed strain history can be expressed as an integral where G is a function that describes the relaxation of the fiber with time (analogous to the creep function, but the opposite concept).

$$F(t) = \int_{-\infty}^{t} G(t-\tau)\frac{d\varepsilon}{dt}d\tau$$

Reference: Oomens, C., Brekelmans, M., & Baaijens, F. (2009). *Biomechanics: Concepts and Computation*. Cambridge University Press.